Toward a universal model of breaking waves on shallow water

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Approximation of the Euler equations

- Non-linear long waves
- Flow structure
Flow structure

The evidence suggests that a strong shear surface layer appears when the wave breaks.
Two-layer flow structure: potential case

Rankine-Hugoniot relations: (Ostapenko (2001), Abgrall & Karni (2009),
SG & Kazakova (2014), ...)

Dispersive case: Barros, Teshukov & SG (2007), Barros and SG (2007),
Percival, Cotter & Holm (2009), ...

Flow structure

$u \neq \nabla \varphi$
Weakly sheared flows

\[ \omega \approx u_z = O(\varepsilon^\beta), \quad \varepsilon = H/L \ll 1. \]

- Large vorticity, hyperbolic model: \(0 < \beta < 1\) (Teshukov 2007, Richard and SG, 2012, 2013)
- Middle vorticity, dispersive model: \(\beta = 1\) (Castro, Lannes, 2014)
- Small vorticity, dispersive model: \(1 < \beta < 2\) (Richard and SG, 2015)

How to find the ‘equilibrium’ between the large vorticity and dispersion?
Flow structure

\[ \mathbf{u} = \eta(t, x) \]

- dispersive potential layer
- interface shear layer
- free surface
- mixing interface
- dispersive potential layer

\[ z = \eta(t, x) \]
\[ z = h(t, x) \]
\[ z = b(x) \]
Euler equations

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x},
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) = -\rho g - \frac{\partial p}{\partial z}.
\]

Classical kinematic and dynamic boundary conditions plus the following condition at the internal boundary:

\[
w \mid_{z=b+h} - \frac{\partial h}{\partial t} - u \mid_{z=b+h} \frac{\partial h}{\partial x} = M.
\]
Upper layer

- large vorticity \( \omega \approx O(\varepsilon^\beta), \quad 0 < \beta < 1 \) (Teshukov, 2007; G. Richard and SG (2012, 2013, 2015)).
- the upper free surface: customary dynamic and kinematic boundary conditions
- the interface between layer: mixing is present.
Upper layer: dimensionless equations for a flat bottom

\[ \eta_t + (\eta \bar{u})_x = M, \]

\[ \frac{\partial}{\partial t} (\eta \bar{u}) + \frac{\partial}{\partial x} \left( \eta \bar{u}^2 + \eta e + \frac{\eta^2}{2} \right) = MU - p_{|z=h} h_x, \]

\[ \left( \eta \left( \frac{\bar{u}^2}{2} + \frac{e}{2} + \frac{\eta + 2h}{2} \right) \right)_t + \left( \eta \bar{u} \left( \frac{\bar{u}^2}{2} + \frac{e}{2} + \frac{\eta + 2h}{2} \right) \right) + \left( \eta e + \frac{\eta^2}{2} \right) \bar{u} \]

\[ = M \left( \frac{U^2}{2} + h \right) + p_{|z=h} (h_t + M) - d, \]

where

\[ p_{|z=h} = \eta. \]
Lower layer

- Potential (almost) flow, second order approximation.
- the interface: customary dynamic condition and a kinematic boundary condition with mixing
- at the bottom: no-penetration condition
Lower layer : dimensionless equations for a flat bottom

\[ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hU) = -M, \]

\[ \frac{\partial}{\partial t} (hU) + \frac{\partial}{\partial x} \left( hU^2 + \int_0^h pdz \right) = p_{|z=h} h_x - MU, \]

\[ \left( h \left( \frac{U^2}{2} + \varepsilon^2 \frac{U_x^2}{6} h^2 + \frac{h}{2} \right) \right)_t + \left( hU \left( \frac{U^2}{2} + \varepsilon^2 \frac{U_x^2}{6} h^2 + \frac{h}{2} \right) + U \int_0^h pdz \right)_x \]

\[ = - p_{|z=h} (h_t + M) - M \left( \frac{U^2}{2} + \varepsilon^2 \frac{U_x^2 h^2}{2} + h \right), \]

where

\[ p_{|z=h} = \eta, \quad \int_0^h pdz = p_{|z=h} h + \frac{h^2}{2} - \frac{\varepsilon^2 h^3}{3} (U_{xt} + UU_{xx} - U_x^2). \]
Mixing

\[ z = -Dx \]

**TURBULENT**

**POTENTIAL**

\[ z = -Dx \]
Closure hypotheses (A. A. Townsend, P. Bradshaw)

\[ M = \sigma \sqrt{e}, \quad \sigma = \frac{-< u'w' >}{< u'^2 > + < w'^2 >} \approx 0.15. \]
Supertank \((300 \, m \times 5 \, m \times 5.2 \, m)\) at THL of National Cheng Kung University, Taiwan
Solitary wave transformation on a mild sloping beach

**Figure** – Shoaling and breaking

- Slope 1/60
- $t^* = 0, 50, 75, \quad t^* = t \sqrt{\frac{g}{h_0}}$. 
Solitary wave transformation: trial 43 \((a/H_0 = 0.152)\)

**Figure** – Shoaling and breaking
Wave breaking in a 300 m channel
Favre waves

**Figure** – Qualitative description of Favre waves
Favre waves, $F = 1.28$
Favre waves, $F = 1.33$
Favre waves, $F = 1.4$
FUTURE WORK

- Modelling of bubble entrainment
- Multi-D modelling of wave breaking